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A Linear Scheme for the Numerical Solution of Nonlinear Quasistationary Magnetic Fields

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Abstract. The computation of nonlinear quasistationary two-dimensional magnetic fields leads to the following problem. There exists a bounded domain Ω and an open nonempty set $R \subset \Omega$. We are looking for the magnetic vector potential $u(x_1, x_2, t)$ which satisfies: (1) a certain nonlinear parabolic equation and an initial condition in R, (2) a nonlinear elliptic equation in $S = \Omega - \bar{R}$, (3) a boundary conditon on $\partial\Omega$ and the condition that u as well as its conormal derivative are continuous across $\Gamma = \partial R \cap \partial S$. This problem is formulated in an abstract variational way. We construct an approximate solution discretized in space by a generalized Galerkin method and by a one-step method in time. The resulting scheme is unconditionally stable and linear. A strong convergence of the approximate solution is proved without any regularity assumptions for the exact solution. We also derive an error bound for the solution of the two-dimensional magnetic field equations under the assumption that the exact solution is sufficiently smooth.

1. Introduction. For two media the computation of nonlinear quasistationary two-dimensional magnetic fields leads to the following problem: There exists a two-dimensional bounded domain Ω and an open nonempty set $R \subset \Omega$. We are looking for the x_3-component $u = u(x_1, x_2, t)$ of the magnetic vector potential such that

$$\sigma \frac{\partial u}{\partial t} = \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(\nu \frac{\partial u}{\partial x_i}\right) + J \quad \text{in } R \times (0, T), 0 < T < \infty, \tag{1.1}$$

$$u(x_1, x_2, 0) = u_0(x_1, x_2) \quad \text{in } R,$$

$$0 = \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(\nu \frac{\partial u}{\partial x_i}\right) + J \quad \text{in } S \times (0, T), \qquad S = \Omega - \bar{R}, \tag{1.2}$$

u satisfies a boundary condition on $\partial\Omega \times (0, T)$ and

$$[u]_R^S = \left[\nu \frac{\partial u}{\partial n}\right]_R^S = 0 \quad \text{on } \Gamma \times (0, T), \qquad \Gamma = \partial R \cap \partial S. \tag{1.3}$$

Here the conductivity $\sigma = \sigma(x_1, x_2)$ is a positive function on R, the reluctivity $\nu = \nu(x_1, x_2, \|\text{grad } u\|_2)$ ($\|\text{grad } u\|_2^2 = \Sigma_{i=1}^2 (\partial u / \partial x_i)^2$) is a positive function on $\Omega \times [0, \infty)$, $J = J(x_1, x_2, t)$ is a given current density, $u_0 = u_0(x_1, x_2)$ is a given initial value of the x_3-component of the magnetic vector potential and n denotes the normal to Γ oriented in a unique way. The derivation of (1.1) and (1.2) from Maxwell's equations is, e.g., given in Demerdash and Gillot [10].

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In Zlámal [9] there are given two equivalent abstract formulations of the above problem. A fully discrete approximate solution is constructed and a weak convergence is proved. The scheme for the approximate solution is nonlinear. In this paper the hypotheses on relevant spaces and differential operators are strengthened. On the other hand, the scheme for the fully discrete approximate solution is unconditionally stable and linear; more exactly the corresponding matrix is the same at each time step. In case of the Dirichlet homogeneous boundary condition it is derived from the equation

$$(1.4)\quad \left(\sigma\frac{\partial u}{\partial t}, v\right)_{L^2(R)} + a(u,v) = (J,v)_{L^2(\Omega)},\ a(u,v) = \int_\Omega \sum_{i=1}^{2} \nu \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}dx,$$

which is true for all $v \in V = H_0^1(\Omega)$ (it follows by multiplying (1.1) and (1.2) by v, by integrating over R and S, respectively, by using Green's theorem, by summing up and by taking into account (1.3)). We add the bilinear form

$$l(u,v) = \sum_{M=R,S} \Theta_M \int_M \sum_{i=1}^{2} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}dx$$

to both sides of (1.4), where Θ_M $(M = R, S)$ are positive constants, and we discretize in space by the Galerkin method. Denoting by U the semidiscrete solution, we get

$$(1.5)\qquad \left(\sigma\frac{\partial U}{\partial t}, v\right)_{L^2(R)} + l(U,v) = \omega(U,v) + (J,v)_{L^2(\Omega)} \quad \forall \in V^h;$$

here $\omega(u,v) = l(u,v) - a(u,v)$, $V^h \to V$ is a family of finite-dimensional approximations of the space V and $U \in V^h$. The discretization in time is carried out by applying the implicit Euler method to the left-hand side and the explicit Euler method to the right-hand side of (1.5). The final scheme is

$$(1.6)\qquad (U^i - U^{i-1}, \sigma v)_{L^2(R)} + \Delta t l(U^i, v)$$
$$= \Delta t\omega(U^{i-1}, v) + \Delta t(J^{i-1}, v)_{L^2(\Omega)} \quad \forall v \in V^h;$$

here the index i denotes the value of the corresponding function at the time $t_i = i\Delta t$, $i = 0, 1, \ldots$. The scheme (1.6) cannot be used for $i = 1$ as the initial value u_0 is known on R only. U^1 has to be computed by the nonlinear scheme (4.4). Let us remark that the idea of implicit-explicit methods goes back to Douglas and Dupont who proposed in [4] the Laplace modified method for the solution of the nonlinear heat equation. Recently, Crouzeix [3] proposed a general scheme of an implicit-explicit linear multistep method. We prove a strong convergence of the approximate solution to the exact one in two norms for the abstract variational formulation of the problem without requiring any smoothness of the exact solution. The conditions of the main theorem are satisfied for the problem (1.1)–(1.3) if the reluctivity ν and the constants Θ_M satisfy (2.4) and (3.1), respectively. Under these conditions we also derive an error bound assuming, of course, that the exact solution is sufficiently smooth. The condition $\Theta_M > \frac{1}{2}C_M$ is almost necessary in the following sense: if $\nu = \text{const}$, then $\Theta_M \geqslant \frac{1}{2}C_M$ is necessary (as well as sufficient) for the scheme (1.6) to be unconditionally stable.

The abstract variational formulation covers the three-dimensional nonlinear quasistationary magnetic field as well. The magnetic vector potential $\mathbf{u}$ is now a vector $\mathbf{u} = (u_1, u_2, u_3)^T$ with $u_i = u_i(x_1, x_2, x_3, t)$ and it satisfies (see [10]):

$$\tag{1.7} \begin{aligned} &\sigma\frac{\partial \mathbf{u}}{\partial t} = -\operatorname{curl}(\nu \operatorname{curl}\mathbf{u}) + \mathbf{J} \quad \text{in } R \times (0, T), \\ &\mathbf{u}(x_1, x_2, x_3, 0) = \mathbf{u}_0(x_1, x_2, x_3) \quad \text{in } R, \end{aligned}$$

$$\tag{1.8} \mathbf{0} = -\operatorname{curl}(\nu \operatorname{curl}\mathbf{u}) + \mathbf{J} \quad \text{in } S \times (0, T), \qquad S = \Omega - \bar{R},$$

$\mathbf{u}$ satisfies a boundary condition on $\partial\Omega \times (0, T)$ and

$$\tag{1.9} \begin{aligned} &\left[(\operatorname{curl}\mathbf{u})^T\mathbf{n}\right]_R^S = 0, \qquad [\nu \operatorname{curl}\mathbf{u} \times \mathbf{n}]_R^S = \mathbf{0} \quad \text{on } \Gamma \times (0, T), \\ &\Gamma = \partial R \cap \partial S. \end{aligned}$$

Here $\mathbf{n}$ is the unit normal vector to Γ oriented in a unique way, $\mathbf{u} \times \mathbf{v}$ and $\mathbf{u}^T\mathbf{v}$ (the superscript T denotes transposition of a vector or of a matrix) denote the vector and the scalar product, respectively, and $\nu = \nu(x_1, x_2, x_3, \|\operatorname{curl}\mathbf{u}\|_2)$. For the boundary condition $\mathbf{u}|_{\partial\Omega} = \mathbf{0}$, the equation corresponding to (1.4) has the form

$$\tag{1.10} \begin{aligned} &\left(\sigma\frac{\partial \mathbf{u}}{\partial t}, \mathbf{v}\right)_{(L^2(R))^3} + a(\mathbf{u}, \mathbf{v}) = (\mathbf{J}, \mathbf{v})_{(L^2(\Omega))^3}, \\ &a(\mathbf{u}, \mathbf{v}) = \int_\Omega \nu(\operatorname{curl}\mathbf{u})^T \operatorname{curl}\mathbf{v}\, dx, \end{aligned}$$

which is true for all $\mathbf{v} \in V = (H_0^1(\Omega))^3$. (1.10) can be derived in the same way as (1.4). Adding the bilinear form

$$l(\mathbf{u}, \mathbf{v}) = \sum_{M=R,S} \Theta_M(\operatorname{curl}\mathbf{u}, \operatorname{curl}\mathbf{v})_{(L^2(M))^3}$$

to both sides of (1.10) and discretizing in space and in time as before, we get the linear scheme

$$\tag{1.11} \begin{aligned} &(\Delta\mathbf{U}^i, \sigma\mathbf{v})_{(L^2(R))^3} + \Delta t\, l(\mathbf{U}^i, \mathbf{v}) \\ &\qquad = \Delta t\, \omega(\mathbf{U}^{i-1}, \mathbf{v}) + \Delta t(\mathbf{J}^{i-1}, \mathbf{v})_{(L^2(\Omega))^3} \quad \forall v \in V^h, \end{aligned}$$

where $V^h \to V$ is a family of finite-dimensional approximations of the space $V = (H^1(\Omega))^3$, $\omega(\mathbf{u}, \mathbf{v}) = l(\mathbf{u}, \mathbf{v}) - a(\mathbf{u}, \mathbf{v})$. Again, the conditions (2.4) and (3.1) are sufficient for applying the abstract convergence result.

2. General Formulation of the Problem. First, we introduce some notations. $H^k(\Omega)$, $k = 0, 1, \dots$, denotes the usual Sobolev space, $H^k(\Omega) = \{v \in L^2(\Omega);\ D^\alpha v \in L^2(\Omega)\ \forall |\alpha| \leq k\}$, provided with the norm $\|v\|_{H^k(\Omega)} = \sum_{|\alpha| \leq k}\|D^\alpha v\|_{L^2(\Omega)}$. $H_0^k(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in the norm $\|\cdot\|_{H^k(\Omega)}$, $H^{-k}(\Omega) = (H_0^k(\Omega))'$ provided with the dual norm. If X is a Banach space normed by $\|\cdot\|_X$ and $p \geq 1$, we denote by $L^p(0, T; X)$, $0 < T < \infty$, the space of strongly measurable functions f: $(0, T) \to X$ such that

$$\|f\|_{L^p(0,T;X)} = \left[\int_0^T \|f(t)\|_X^p\, dt\right]^{1/p} < \infty,$$

with the usual $p = \infty$ modification. By $C([0, T]; X)$ we denote the space of continuous functions f: $[0, T] \to X$ normed by $\|f\|_{C([0,T];X)} = \max_{t \in [0,T]}\|f(t)\|_X$, and by $C^{0,1}([0,T]; X)$ the space of Lipschitz continuous functions. If $u \in L^1(0,T; X)$, we denote by u' the weak or generalized derivative of u (see Temam [8, Lemma 1.1, p. 250]).

Now, we introduce three requirements concerning relevant spaces, differential operators and data.

A. Let H_M, $M = R, S$, be two (real) Hilbert spaces with scalar products $(\cdot,\cdot)_M$ (the induced norms are denoted by $|\cdot|_M$), and let the Hilbert space $H = H_R \times H_S$ (with elements $[v_R, v_S]$, $v_R \in H_R$, $v_S \in H_S$) have the scalar product $(\cdot,\cdot)$ such that the norm $|v| = (v,v)^{1/2}$ is equivalent with $|v_R|_R + |v_S|_S$. Further, let $V \subset H$ be a separable Hilbert space normed by $\|\cdot\|$, and let the vector spaces $V_M = \{\omega \mid \omega = v_M, v \in V\}$ be subspaces of the Hilbert spaces $B_M \subset H_M$ normed by $\|\cdot\|_M$. Let $((\cdot,\cdot))_{1,M}$ be bilinear symmetric positive (not necessarily definite) forms on $\overline{V}_M \times \overline{V}_M$, $\overline{V}_M$ being the closure of V_M in B_M. Then $\|v\|_1 = [\|v_R\|_{1,R}^2 + \|v_S\|_{1,S}^2]^{1/2}$ with $\|v_M\|_{1,M}^2 = ((v_M, v_M))_{1,M}$ is a seminorm on $V \times V$. We require that

$$\left.\begin{aligned} &\|v_M\|_{1,M} \leqslant C\|v_M\|_M, && \|v_R\|_R + \|v_S\|_S \leqslant C\|v\|, \\ &\|v\|_1 + \lambda|v_R|_R \geqslant \beta\|v\|, && C, \lambda, \beta = \text{const} > 0 \end{aligned}\right\} \quad \forall v \in V. \tag{2.1}$$

We denote by $\mathring{V}_R$ the closed subspace $\{\omega \mid \omega = v_R, v \in V, v_S = 0\}$ of $\overline{V}_R$, and we assume $\mathring{V}_R$ to be dense in H_R and $\overline{V}_R$ to be compactly imbedded in H_R.

Example 1. Let Ω, R, S be domains introduced in Section 1 with Lipschitz boundaries. We choose $H_M = L^2(M)$, $(u,v)_R = (\sigma u, v)_{L^2(R)}$, where $\sigma \in L^\infty(R)$, $\sigma \geqslant \sigma_0 > 0$,

$$(u,v)_S = (u,v)_{L^2(S)}; \quad H = L^2(\Omega), \quad (u,v) = (u,v)_{L^2(\Omega)}, \quad V = H_0^1(\Omega),$$

$$\overline{V}_M = \{\omega \mid \omega = H^1(M), \omega|_{\partial\Omega\cap\partial M} = 0\}, \qquad \|v\|_M = \|v\|_{H^1(M)},$$

$$((u,v))_{1,M} = \int_M \sum_{i=1}^2 \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,dx, \qquad \|u\|_1 = \left\{\int_\Omega \sum_{i=1}^2 \left(\frac{\partial u}{\partial x_i}\right)^2 dx\right\}^{1/2}.$$

In three dimensions we take

$$H_M = (L^2(M))^3, \quad (u,v)_R = (\sigma\mathbf{u},\mathbf{v})_{(L^2(R))^3}, \quad (u,v)_S = (\mathbf{u},\mathbf{v})_{(L^2(S))^3},$$

$$H = (L^2(\Omega))^3, \quad (u,v) = (\mathbf{u},\mathbf{v})_{(L^2(\Omega))^3}, \quad V = (H_0^1(\Omega))^3,$$

$$\overline{V}_M = \{\omega \mid \omega \in (H(M))^3; \omega|_{\partial\Omega\cap\partial M} = \mathbf{0}\}, \qquad \|v\|_M = \|\mathbf{v}\|_{(H^1(M))^3},$$

$$((u,v))_{1,M} = (\operatorname{curl}\mathbf{u}, \operatorname{curl}\mathbf{v})_{(L^2(M))^3}, \quad \|u\|_1 = \|\operatorname{curl}\mathbf{u}\|_{(L^2(\Omega))^3}.$$

As $\mathring{V}_R$ is dense in H_R, $\overline{V}_R$ is also dense in H_R. We identify H_R with its dual space by means of its scalar product $(\cdot,\cdot)_R$. From the continuous imbedding of $\overline{V}_R$ into H_R it follows that H_R can be identified with subspaces of $\mathring{V}_R$ and of $\overline{V}_R$, and we have inclusions $\overline{V}_R \subset H_R \subset \overline{V}'_R$, $\mathring{V}_R \subset H_R \subset \mathring{V}'_R$ where each space is dense in the following one and the injections are continuous. Furthermore, the scalar product $\langle\cdot,\cdot\rangle_R$ in the duality between $\overline{V}'_R$ and $\overline{V}_R$ is an extension of $(\cdot,\cdot)_R$, i.e.

$$\langle u, v\rangle_R = (u,v)_R \quad \text{if } u \in H_R, v \in \overline{V}_R.$$

We denote the scalar product between V' and V by $\langle\cdot,\cdot\rangle$ and between $\overline{V}'_S$ and $\overline{V}_S$ by $\langle\cdot,\cdot\rangle_S$.

B. Let $A^M(\varphi)$, $M = R, S$, be gradients of the functionals $J^M(\varphi)$ defined on $\bar{V}_M$ and having Hessians $H^M(w)$. Further, let

$$J^M(0) = 0, \qquad A^M(0) = 0 \tag{2.2}$$

and

$$\begin{cases} \left|\langle H^M(w)\varphi, \psi\rangle_M\right| \leq C_M \|\varphi\|_{1,M}\|\psi\|_{1,M} \\ \langle H^M(w)\varphi, \varphi\rangle_M \geq c_M\|\varphi\|^2_{1,M} \end{cases} \forall w, \varphi, \psi \in \bar{V}_M, \tag{2.3}$$
$$\langle H^S(w)\varphi, \varphi\rangle_S > 0 \quad \forall w \in \bar{V}_S,$$
$$\forall \varphi \in \mathring{V}_S = \{\omega \mid \omega = v_S,\, v \in V,\, v_R = 0\},\, \varphi \neq 0,\, 0 < c_M \leq C_M < \infty.$$

C. Let f^M, $M = R, S$, be functionals from $C^{0,1}([0, T]; \bar{V}'_M)$ and $u_0 \in H_R$.

Example 2. We consider the spaces of Example 1 and the problems (1.1)–(1.3) and (1.7)–(1.9). In applications, $\nu(x_1, x_2, \xi)$ is of the form $\nu(x_1, x_2, \xi) = \nu_M(\xi)$ on M with $\nu_M(\xi) \in C^1([0, \infty))$. Then

$$A^M(\varphi) = -\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left[\nu_M(\|\text{grad } \varphi\|_2)\frac{\partial \varphi}{\partial x_i}\right],$$

$$\langle A^M(\varphi), \psi\rangle_M = \int_M \sum_{i=1}^{2} \nu_M(\|\text{grad } \varphi\|_2)\frac{\partial \varphi}{\partial x_i}\frac{\partial \psi}{\partial x_i}\,dx,$$

$$J^M(\varphi) = \int_M F_M(\|\text{grad } \varphi\|_2)\,dx,$$

where $F_M(\xi) = \int_0^\xi s\nu_M(s)\,ds$. In three dimensions

$$A^M(\boldsymbol{\varphi}) = \text{curl}\big(\nu_M(\|\text{curl } \boldsymbol{\varphi}\|_2)\text{curl } \boldsymbol{\varphi}\big),$$

$$\langle A^M(\boldsymbol{\varphi}), \boldsymbol{\psi}\rangle_M = \int_M \nu_M(\|\text{curl } \boldsymbol{\varphi}\|_2)(\text{curl } \boldsymbol{\varphi})^T \text{curl } \boldsymbol{\psi}\,dx,$$

$$J^M(\boldsymbol{\varphi}) = \int_M F_M(\|\text{curl}\boldsymbol{\varphi}\|_2)\,dx.$$

We shall prove in the last section that B is satisfied if

$$c_M \leq \frac{d}{d\xi}[\xi\nu_M(\xi)] \leq C_M \quad \forall \xi \in [0, \infty). \tag{2.4}$$

The last notations which we need are

$$W_R = \{u \mid u \in L^2(0, T; V);\ u'_R \in L^2(0, T; \bar{V}'_R)\},$$
$$a(u, v) = \langle A^R(u_R), v_R\rangle_R + \langle A^S(u_S), v_S\rangle_S \quad \forall u, v \in V,$$
$$\langle f, v\rangle = \langle f^R, v_R\rangle_R + \langle f^S, v_S\rangle_S \quad \forall v \in V.$$

The abstract problem in which we are interested can be formulated in two equivalent (see [9]) ways as follows: Find $u \in W_R$ such that

$$u'_R + A^R(u_R) = f^R, \quad u(0)_R = u_0, \quad A^S(u_S) = f^S \tag{P}$$

or

$$\frac{d}{dt}(u_R, v_R)_R + a(u, v) = \langle f, v\rangle \quad \text{in } \mathcal{D}'((0, T)) \quad \forall v \in V, \quad u(0)_R = u_0. \tag{P'}$$

If the condition A is satisfied then it is easy to see that the condition 1) of Theorems 1 and 2 of [9] is fulfilled. We shall later show that if also B is satisfied, then the assumptions 2, 3, 4, 5 of the mentioned theorems are fulfilled (with $p = 2$ and $[v] = \|v\|_1$). Therefore, the problems (P) and (P′) are equivalent and there exists just one solution of these problems.

3. Approximate Solution, Convergence. To define the approximate solution we discretize (P′) in space and in time. The discretization in space is carried out by means of a generalized Galerkin method (see Nečas [7, p. 47]). To this end we assume that there exists a family $\{V^h\}$, $0 < h < h^*$, $h^* > 0$, of finite-dimensional subspaces of V such that

D. $\lim_{h\to 0+} \operatorname{dist}(V^h, v) = 0 \ \forall v \in V$

(see three remarks following Eq. (3.22) in [9]). We also consider a partition $0 = t_0 < t_1 < t_2 < t_r = T$ of the interval $[0, T]$, where $t_i = i\Delta t$, $i = 0, \ldots, r$, $\Delta t = T/r$. We choose the constants Θ_M to satisfy

$$\Theta_M > \tfrac{1}{2} C_M. \tag{3.1}$$

We denote

$$\begin{cases} l(u, v) = \Theta_R((u_R, v_R))_{1,R} + \Theta_S((u_S, v_S))_{1,S}, \\ \omega(u, v) = l(u, v) - a(u, v) \end{cases} \tag{3.2}$$

and we write (P′) in the form

$$\frac{d}{dt}(u_R, v_R)_R + l(u, v) = \omega(u, v) + \langle f, v \rangle \quad \text{in } \mathcal{D}'((0, T)) \ \forall v \in V, \qquad u(0)_R = u_0. \tag{3.3}$$

Discretizing in space and integrating the left-hand side of (3.3) by the Euler implicit method and the right-hand side by the Euler explicit method, we get a scheme which is linear:

$$\begin{cases} (U_R^i - U_R^{i-1}, v_R)_R + \Delta t\, l(U^i, v) = \Delta t\, \omega(U^{i-1}, v) + \Delta t \langle f^{i-1}, v \rangle \\ \qquad\qquad \forall v \in V^h, i \geq 2. \end{cases} \tag{3.4}$$

The existence and uniqueness of $U^i \in V^h$ follows from the fact that the quadratic form $(v_R, v_R)_R + \Delta t\, l(v, v)$ is bounded from below by $c\Delta t\|v\|^2$ (in the sequel, c and C denote generic positive constants which do not depend on $\delta = (h, \Delta t)$ and which are not necessarily the same at any two places). The boundedness is a consequence of the inequality

$$|v_R|_R^2 + C_1 \Delta t \|v\|_1^2 \geq c_1(C_1, \beta)\Delta t \|v\|^2 \quad \forall v \in V, \tag{3.5}$$

which is true for any $C_1 > 0$ and for $\Delta t \leq c_2(\lambda, C_1)$ owing to the third inequality of (2.1).

In (3.4) we cannot choose $i = 1$ as $u(0)_S$ is not given. Therefore, U^1 must be defined in a different way. Before doing this we introduce the last assumption.

E. The initial value u_0 belongs to V_R, i.e. there exists $g \in V$ such that $u_0 = g_R$. Evidently, if E is satisfied, then from D it follows that there exists $g^h \in V^h$ such that $|u_0 - g_R^h|_R \to 0$, $\|g^h\| \leq C$ for $0 < h < h^*$.

U^1 is defined as follows:

$$(3.6)\quad (U_R^1 - U_R^0, v_R)_R + \Delta t a(U^1, v) = \Delta t \langle f^1, v \rangle \quad \forall v \in V^h,\ U_R^0 = g_R^h.$$

(3.6) is a nonlinear scheme considered (for arbitrary $i \geqslant 1$) in [9]. Hence, the existence and uniqueness of U^1 follows from Theorem 2 of [9].

Remark. In computations we do not need to know the extension g of u_0 which one can see at first glance from (3.6). We need g_R^h only, and we can choose $g_R^h =$ the interpolate of u_0.

We extend the approximate solution on the whole interval $[0, T]$:

$$(3.7)\quad \begin{gathered} U^\delta = U^{i-1} + \frac{t - t_{i-1}}{\Delta t}(U^i - U^{i-1}) \quad \text{in } [t_{i-1}, t_i],\ i = 1, \ldots, r, \\ U^0 = g^h,\ \delta = (h, \Delta t). \end{gathered}$$

Evidently $U^\delta \in C([0, T]; V)$.

THEOREM 3.1. *Under the conditions* A, B, C, D, E *and* (3.1) *the approximate solution* U^δ *is uniquely determined by* (3.4), (3.6) *and* (3.7), *it belongs to* $C([0, T]; V)$ *and*

$$(3.8)\quad \|u_R - U_R^\delta\|_{C([0,T]; H_R)} \to 0, \quad \|u - U^\delta\|_{L^2(0,T;V)} \to 0 \quad \text{if } \delta \to 0;$$

here u is the unique solution of the problem (P′) *and* $u \in L^\infty(0, T; V)$.

Proof. (a) We use the compactness method (see Lions [6] and the references given there). We show that from any sequence $\{U^{\delta_j}\}$ of the family $\{U^\delta\}$ with $\delta_j \to 0$ one can choose a subsequence $U^{\delta_{j(\nu)}}$ such that

$$\delta_{j(\nu)} \to 0, \quad \|u_R - U_R^{\delta_{j(\nu)}}\|_{C([0,T]; H_R)} \to 0, \quad \|u - U^{\delta_{j(\nu)}}\|_{L^2(0,T,V)} \to 0$$

and that u is a solution of the problem (P′). As (P′) has a unique solution, (3.8) follows.

(b) From Taylor's formula (see, e.g. Céa [1, p. 52]) and from (2.2) it follows that

$$J^M(\varphi) = \tfrac{1}{2}\langle H^M(\vartheta\varphi)\varphi, \varphi \rangle_M, \qquad 0 < \vartheta < 1.$$

Hence by (2.3)

$$\tfrac{1}{2}c_M\|\varphi\|_{1,M}^2 \leqslant J^M(\varphi) \leqslant \tfrac{1}{2}C_M\|\varphi\|_{1,M}^2.$$

Setting $J(v) = J^R(v_R) + J^S(v_S)$, we see that

$$(3.9)\quad \begin{gathered} \tfrac{1}{2}c_0\|v\|_1^2 \leqslant J(v) \leqslant \tfrac{1}{2}C_0\|v\|_1^2 \quad \forall v \in V, \\ c_0 = \min(c_R, c_S) > 0, \qquad C_0 = \max(C_R, C_S). \end{gathered}$$

Further,

$$a^M(\varphi, \psi) = \langle A^M(\varphi), \psi \rangle_M = \langle H^M(\vartheta\varphi)\varphi, \psi \rangle_M, \qquad 0 < \vartheta < 1.$$

Therefore,

$$|a^M(\varphi, \psi)| \leqslant C_M\|\varphi\|_{1,M}\|\psi\|_{1,M}, \qquad a^M(\varphi, \varphi) \geqslant c_M\|\varphi\|_{1,M}^2.$$

Also

$$a^M(\varphi, \omega) - a^M(\psi, \omega) = \langle H^M(\psi + \vartheta(\varphi - \psi))(\varphi - \psi), \omega \rangle_M.$$

Thus

$$|a^M(\varphi, \omega) - a^M(\psi, \omega)| \leq C_M \|\varphi - \psi\|_{1,M} \|\omega\|_{1,M}$$

and

$$a^M(\varphi, \varphi - \psi) - a^M(\psi, \varphi - \psi) \geq c_M \|\varphi - \psi\|_{1,M}^2,$$

so that $a^M(\varphi, \psi)$ are monotone and $a^S(\varphi, \psi)$ fulfils (3.7) of [9]. It follows that

$$(3.10) \qquad \begin{cases} |a(u, v)| \leq C_0 \|u\|_1 \|v\|_1, \quad a(u, v) \geq c_0 \|u\|_1^2, \\ a(u, u - v) - a(v, u - v) \geq c_0 \|u - v\|_1^2, \\ |a(u, w) - a(v, w)| \leq C_0 \|u - v\|_1 \|w\|_1 \quad \forall u, v, w \in V. \end{cases}$$

Obviously, the assumptions 2, 3, 4, 5 of Theorems 1 and 2 of [9] (with $p = 2$ and $[v] = \|v\|_1$) are satisfied. Further,

$$a^M(\varphi, \psi - \varphi) + J^M(\varphi) - J^M(\psi) = \langle A^M(\varphi), \psi - \varphi \rangle + J^M(\varphi) - J^M(\psi)$$
$$= -\tfrac{1}{2} \langle H^M(\varphi + \vartheta(\psi - \varphi))(\psi - \varphi), \psi - \varphi \rangle_M, \qquad 0 < \vartheta < 1.$$

Hence

$$a^M(\varphi, \psi - \varphi) + J^M(\varphi) - J^M(\psi) \geq -\tfrac{1}{2} C_M \|\varphi - \psi\|_{1,M}^2,$$

so that

$$(3.11) \quad a(v, u - v) + J(v) - J(u) \geq -\tfrac{1}{2}\left[C_R \|u_R - v_R\|_{1,R}^2 + C_S \|u_S - v_S\|_{1,S}^2\right].$$

We set

$$(3.12) \qquad \kappa_M = \Theta_M - \tfrac{1}{2} C_M.$$

We have $\kappa_M > 0$ owing to (3.1), hence

$$(3.13) \qquad \kappa_0 = \min(\kappa_R, \kappa_S) > 0.$$

From (3.11) it follows that

$$(3.14) \quad l(u - v, u - v) + a(v, u - v) \geq \kappa_0 \|u - v\|_1^2 + J(u) - J(v) \quad \forall u, v \in V.$$

(c) First, we prove that

$$(3.15) \qquad \|U^1\| \leq C.$$

Choosing $v = U^1 - g^h$ in (3.6), we get (using the inequality $ab \leq \frac{1}{2}\vartheta a^2 + \frac{1}{2}\vartheta^{-1} b^2$, $\vartheta > 0$),

$$|U_R^1 - g_R^h|_R^2 + \Delta t a(U^1 - g^h, U^1 - g^h)$$
$$= \Delta t\left[a(U^1 - g^h, U^1 - g^h) - a(U^1, U^1 - g^h)\right]$$
$$+ C\vartheta^{-1}\Delta t + C\vartheta\Delta t \|U^1 - g^h\|^2, \qquad \vartheta > 0.$$

From (3.10) and from the assumption E it follows that

$$|U_R^1 - g_R^h|_R^2 + c_0 \Delta t \|U^1 - g^h\|_1^2 \leq \tfrac{1}{2} c_0 \Delta t \|U^1 - g^h\|_1^2 + C\vartheta^{-1}\Delta t + C\vartheta\Delta t \|U^1 - g^h\|^2.$$

By (3.5)

$$c_1 \Delta t \|U^1 - g^h\|^2 \leq C\vartheta^{-1}\Delta t + C_2 \vartheta \Delta t \|U^1 - g^h\|^2.$$

Setting $\vartheta = c_1/2C_2$, we get $\|U^1 - g^h\|^2 \leqslant C$. (3.15) is a consequence of this estimate and of the assumption E.

Now, we prove the following inequality:

$$\sum_{i=2}^{m} \left|\Delta U_R^i\right|_R^2 + c\Delta t \sum_{i=2}^{m} \left\|\Delta U^i\right\|^2 + c\Delta t\left\|U^m\right\|^2 \leqslant C\Delta t + C\Delta t^2 \sum_{i=2}^{m} \left\|U^i\right\|^2, \qquad 2 \leqslant m \leqslant r; \tag{3.16}$$

here, $\Delta U^i = U^i - U^{i-1}$ and the positive constants c, C do not depend on δ and on m. We choose $v = \Delta U^i$ in (3.4), write this equation in the form

$$\left|\Delta U_R^i\right|_R^2 + \Delta t l(\Delta U^i, \Delta U^i) + \Delta t a(U^{i-1}, \Delta U^i) = \Delta t \left\langle f^{i-1}, \Delta U^i \right\rangle,$$

and we use (3.14). We get

$$\left|\Delta U_R^i\right|_R^2 + \kappa_0 \Delta t\left\|\Delta U^i\right\|_1^2 + \Delta t\left[J(U^i) - J(U^{i-1})\right] \leqslant \Delta t \left\langle f^{i-1}, \Delta U^i \right\rangle.$$

Applying (3.5) to $\frac{1}{2}\{|\Delta U_R^i|_R^2 + 2\kappa_0 \Delta t\|\Delta U^i\|_1^2\}$, adding to both sides $(\Delta U_R^i, U_R^i)_R$ and summing up, we get easily

$$\frac{1}{2}\sum_{i=2}^{m} \left|\Delta U_R^i\right|_R^2 + c_1\Delta t \sum_{i=2}^{m} \left\|\Delta U^i\right\|^2 + \Delta t\left[\tfrac{1}{2}\left|U_R^m\right|_R^2 + J(U^m) - \tfrac{1}{2}\left|U_R^1\right|_R^2 - J(U^1)\right] \leqslant \Delta t \sum_{i=2}^{m} \left(\Delta U_R^i, U_R^i\right)_R + \Delta t \sum_{i=2}^{m} \left\langle f^{i-1}, \Delta U^i \right\rangle.$$

By (3.9), (3.5) and (3.15)

$$\tfrac{1}{2}\left|U_R^m\right|_R^2 + J(U^m) \geqslant c\left\|U^m\right\|^2, \qquad \tfrac{1}{2}\left|U_R^1\right|_R^2 + J(U^1) \leqslant C\left\|U^1\right\|^2 \leqslant C.$$

Estimating the first term on the right-hand side by

$$\frac{1}{4}\sum_{i=2}^{m} \left|\Delta U_R^i\right|_R^2 + C\Delta t^2 \sum_{i=2}^{m} \left\|U^i\right\|^2,$$

we come to the inequality

$$\sum_{i=2}^{m} \left|\Delta U_R^i\right|_R^2 + c\Delta t \sum_{i=2}^{m} \left\|\Delta U^i\right\|^2 + c_2\Delta t\left\|U^m\right\|^2 \leqslant C\Delta t^2 \sum_{i=2}^{m} \left\|U^i\right\|^2 + C\Delta t \sum_{i=2}^{m} \left\langle f^{i-1}, \Delta U^i \right\rangle. \tag{3.17}$$

As

$$\Delta t \sum_{i=2}^{m} \left\langle f^{i-1}, \Delta U^i \right\rangle = -\Delta t\left\langle f^1, U^1 \right\rangle + \Delta t\left\langle f^{m-1}, U^m \right\rangle - \Delta t \sum_{i=2}^{m-1} \left\langle \Delta f^i, U^i \right\rangle$$

and $f \in C^{0,1}([0, T]; V')$, we have

$$\Delta t \sum_{i=2}^{m} \left\langle f^{i-1}, \Delta U^i \right\rangle \leqslant C\vartheta^{-1}\Delta t + C_2\vartheta\Delta t\left\|U^m\right\|^2 + C\Delta t^2 \sum_{i=2}^{m-1} \left\|U^i\right\|^2.$$

Setting $\vartheta = c_2/2C_2$, we arrive at (3.16).

(d) The first consequence of (3.16) is that $\|U^m\|^2 \leq C + C\Delta t \Sigma_{i=2}^m \|U^i\|^2$. By the discrete Gronwall inequality and by (3.15) $\|U^m\| \leq C$ for $1 \leq m \leq r$, hence

$$\|U^\delta(t)\| \leq C \quad \forall t \in [0, T]. \tag{3.18}$$

Further,

$$\Delta t \sum_{i=2}^{m} \|\Delta U^i\|^2 \leq C\Delta t \tag{3.19}$$

and $\Sigma_{i=2}^m |\Delta U_R^i|_R^2 \leq C\Delta t$. This does not mean anything other than

$$\int_0^T \left| \frac{d}{dt} U_R^\delta \right|_R^2 dt \leq C. \tag{3.20}$$

We shall consider sequences of functions from the family $\{U^\delta\}$ and their subsequences. We shall leave out subscripts and use always the same notation $\{U^\delta\}$ for these subsequences, and always δ is such that $\delta \to 0$.

From (3.18) and from the well-known compactness theorem (see, e.g., Céa [1, p. 26]) it follows that there exists a subsequence $\{U^\delta\}$ and an element $u \in L^\infty(0, T; V)$ such that

$$U^\delta \to u \quad \text{in } L^\infty(0, T; V) \text{ weakly*}. \tag{3.21}$$

From (3.20) it follows that

$$|U_R^\delta(t_2) - U_R^\delta(t_1)|_R = \left| \int_{t_1}^{t_2} \frac{d}{dt} U_R^\delta(t)\, dt \right|_R \leq C|t_2 - t_1|^{1/2} \quad \forall t_1, t_2 \in [0, T];$$

therefore the sequence $U_R^\delta(t)$ is equicontinuous on $[0, T]$ in the norm $|\cdot|_R$. Owing to (3.18) it is bounded in $\bar{V}_R$, $\|U_R^\delta(t)\|_R \leq C\ \forall t \in [0, T]$. As the imbedding of $\bar{V}_R$ into H_R is compact (see A) the set $\{U_R^\delta(t)\}$ is relatively compact in H_R for any $t \in [0, T]$. According to the generalization of the Arzelà-Ascoli theorem (see, e.g., Kufner, John, Fučik [5, p. 42]) there exists a subsequence U_R^δ such that $\|\omega - U_R^\delta\|_{C([0,T]; H_R)} \to 0$, where $\omega \in C([0, T]; H_R)$. With regard to (3.21) $\omega = u_R$, i.e.

$$\|u_R - U_R^\delta\|_{C([0,T]; H_R)} \to 0. \tag{3.22}$$

Further, $a(U^\delta, \cdot) \in V'$, and if we denote it by $\langle \chi^\delta, \cdot \rangle$, we conclude from (3.10) and (3.18) that $\chi^\delta \in L^\infty(0, T; V')$ and $\|\chi^\delta\|_{L^\infty(0,T;V')} \leq C$. Similarly, $a^M(U_M^\delta, \cdot) \in \bar{V}_M'$ and denoting it by $\langle \chi^{M,\delta}, \cdot \rangle_M$, we find $\chi^{M,\delta} \in L^\infty(0, T; \bar{V}_M')$ and $\|\chi^{M,\delta}\|_{L^\infty(0,T;\bar{V}_M')} \leq C$. Therefore, there exist subsequences $\{\chi^\delta\}$, $\{\chi^{M,\delta}\}$ such that

$$\begin{cases} \chi^\delta \to \chi \quad \text{in } L^\infty(0, T; V') \text{ weakly*}, \chi \in L^\infty(0, T; V'), \\ \chi^{M,\delta} \to \chi^M \quad \text{in } L^\infty(0, T; \bar{V}_M') \text{ weakly*}, \chi^M \in L^\infty(0, T, \bar{V}_M'). \end{cases} \tag{3.23}$$

(e) Consider a function $h(t) \in \mathfrak{D}((0, T))$, and let us define the function

$$h_{\Delta t} = h^i \quad \text{in } (t_{i-1}, t_i], \qquad i = 1, \dots, r, h^i = h(t_i).$$

For a given $z \in V$ we choose $\{z^h\}$ such that $z^h \in V^h$, $\|z - z^h\| \to 0$, we set $v = z^h h^i$ in (3.4), $v = z^h h^1$ in (3.6) and we sum up. We get

$$\sum_{i=1}^{r} (\Delta U_R^i, z_R^h)_R h^i + \Delta t a(U^1, z^h) h^1 + \Delta t \sum_{i=2}^{r} a(U^{i-1}, z^h) h^i$$
$$= -\Delta t \sum_{i=2}^{r} l(\Delta U^i, z^h) h^i + \Delta t \langle f^1, z^h \rangle h^1 + \Delta t \sum_{i=2}^{r} \langle f^{i-1}, z^h \rangle h^i. \tag{3.24}$$

Concerning the first term on the left-hand side we have

$$\sum_{i=1}^{r}\left(\Delta U_R^i, z_R^h\right)_R h^i = \int_0^T \left(\frac{d}{dt} U_R^\delta, z_R^h\right)_R h_{\Delta t}\, dt$$

$$= \int_0^T \left(\frac{d}{dt} U_R^\delta, z_R\right)_R h\, dt + \int_0^T \left(\frac{d}{dt} U_R^\delta, z_R^h - z_R\right)_R h\, dt$$

$$+ \int_0^T \left(\frac{d}{dt} U_R^\delta, z_R^h\right)_R (h_{\Delta t} - h)\, dt.$$

The last two terms converge to zero and

$$\int_0^T \left(\frac{d}{dt} U_R^\delta, z_R\right)_R h\, dt = -\int_0^T (U_R^\delta, z_R)_R h'\, dt \to -\int_0^T (u_R, z_R)_R h'\, dt.$$

Further,

$$\Delta t a(U^1, z^h)h^1 + \Delta t \sum_{i=2}^{r} a(U^{i-1}, z^h)h^i = \int_0^T a(U^\delta, z^h)h_{\Delta t}\, dt + R,$$

where

$$R = h^1 \int_0^{t_1}\left\{a(U^1, z^h) - a\left(U^0 + \frac{t}{\Delta t}\Delta U^1, z^h\right)\right\} dt$$

$$+ \sum_{i=2}^{r} \int_{t_{i-1}}^{t_i} \left\{a(U^{i-1}, z^h) - a\left(U^{i-1} + \frac{t - t_{i-1}}{\Delta t}\Delta U^i, z^h\right)\right\} h_{\Delta t}\, dt \to 0$$

owing to (3.10), E, (3.15) and (3.19). As

$$\int_0^T a(U^\delta, z^h)h_{\Delta t}\, dt = \int_0^T \langle \chi^\delta, z^h\rangle h_{\Delta t}\, dt \to \int_0^T \langle \chi, z\rangle h\, dt,$$

the left-hand side of (3.24) converges to

$$-\int_0^T (u_R, z_R)_R h'\, dt + \int_0^T \langle \chi, z\rangle h\, dt.$$

From (3.19) and from $f \in C^{0,1}([0, T]; V')$ it is easy to prove that the limit of the right-hand side of (3.24) is $\int_0^T \langle f, z\rangle h\, dt$. Therefore, we have

$$\frac{d}{dt}(u_R, z_R)_R + \langle \chi, z\rangle = \langle f, z\rangle \quad \text{in } \mathcal{D}'((0, T))\ \forall z \in V. \tag{3.25}$$

In the same way as in [9] (see the text following (3.50)) we prove that $u \in W_R$, $u_R' + \chi^R = f^R$, $u(0)_R = u_0$, $\chi^S = f^S$ and that

$$\int_0^T \langle \chi, u\rangle\, dt = \tfrac{1}{2}|u_0|_R^2 - \tfrac{1}{2}|u(T)_R|_R^2 + \int_0^T \langle f, u\rangle\, dt. \tag{3.26}$$

(f) We choose $v = U^1$ in (3.6) and $v = U^i$ in (3.4). Summing up we get

$$\sum_{i=1}^{r}\left(\Delta U_R^i, U_R^i\right)_R + \Delta t\left[a(U^1, U^1) + \sum_{i=2}^{r} a(U^{i-1}, U^i)\right]$$

$$= -\Delta t \sum_{i=2}^{r} l(\Delta U^i, U^i) + \Delta t\left[\langle f^1, U^1\rangle + \sum_{i=2}^{r}\langle f^{i-1}, U^i\rangle\right].$$

Now, taking into account (3.20) and (3.22), we see that

$$\sum_{i=1}^{r} (\Delta U_R^i, U_R^i)_R = \tfrac{1}{2}|U^\delta(T)_R|_R^2 - \tfrac{1}{2}|U^\delta(0)_R|_R^2 + \tfrac{1}{2}\sum_{i=1}^{r}|\Delta U_R^i|_R^2$$
$$\to \tfrac{1}{2}|u(T)_R|_R^2 - \tfrac{1}{2}|u_0|_R^2.$$

Further, it is easy to show that

$$\Delta t\left[a(U^1, U^1) + \sum_{i=2}^{r} a(U^{i-1}, U^i)\right] - \int_0^T a(U^\delta, U^\delta)\,dt \to 0,$$

$$-\Delta t \sum_{i=2}^{r} l(\Delta U^i, U^i) + \Delta t\left[\langle f^1, U^1\rangle + \sum_{i=2}^{r}\langle f^{i-1}, U^i\rangle\right] \to \int_0^T \langle f, u\rangle\,dt.$$

Thus, with regard to (3.26)

$$\text{(3.27)}\quad \int_0^T a(U^\delta, U^\delta)\,dt \to \tfrac{1}{2}|u_0|_R^2 - \tfrac{1}{2}|u(T)_R|_R^2 + \int_0^T \langle f, u\rangle\,dt = \int_0^T \langle \chi, u\rangle\,dt.$$

From (3.21), (3.23) and (3.27) it follows that

$$\lim_{\delta\to 0}\int_0^T [a(u, u - U^\delta) - a(U^\delta, u - U^\delta)]\,dt = 0.$$

As $a(u, u - U^\delta) - a(U^\delta, u - U^\delta) \geqslant c_0\|u - U^\delta\|_1^2$ (see (3.10)), we have

$$\lim_{\delta\to 0}\int_0^T \|u - U^\delta\|_1^2\,dt = 0.$$

(3.22) gives

$$\lim_{\delta\to 0}\int_0^T |u_R - U_R^\delta|_R^2\,dt = 0,$$

and by means of the last assumption in (2.1) we get (3.8). Also, we have

$$\left|\int_0^T [a(u, v) - \langle \chi^\delta, v\rangle]\,dt\right| = \left|\int_0^T [a(u, v) - a(U^\delta, v)]\,dt\right|$$
$$\leqslant C_0\int_0^T \|u - U^\delta\|_1\|v\|_1\,dt \to 0,$$

i.e.

$$\int_0^T [a(u, v) - \langle \chi, v\rangle]\,dt = 0 \quad \forall v \in L^\infty(0, T; V).$$

Setting $v = zh(t)$, $z \in V$, $h \in \mathfrak{D}((0, T))$, we get $\langle \chi, z\rangle = a(u, z)$ $\forall z \in V$, hence u is the solution of the problem (P′) (see (3.25)) and the proof is finished.

4. Nonlinear Magnetic Field. We consider the problem (1.1)–(1.3), and we specify the boundary condition:

$$\text{(4.1)}\qquad u = 0 \quad \text{on } \partial\Omega \times (0, T).$$

We assume that $\partial\Omega$ and ∂R are polygons. The condition A is satisfied, and $\|\cdot\|_1$ is a norm on $V = H_0^1(\Omega)$ (see Example 1). Let $\nu(x_1, x_2, \xi)$, σ and the operators $A^M(\varphi)$, $M = R, S$, be of the form introduced in Example 2. We also assume (2.4) to be fulfilled, and we want to prove that condition B is satisfied. We have

$$\langle H^M(w)\varphi, \psi\rangle_M = \int_M \left\{ \nu_M(\xi) \sum_{i=1}^{2} \frac{\partial\varphi}{\partial x_i}\frac{\partial\psi}{\partial x_i} + \xi^{-1}\nu_M'(\xi) \sum_{i=1}^{2} \frac{\partial w}{\partial x_i}\frac{\partial\varphi}{\partial x_i} \sum_{j=1}^{2} \frac{\partial w}{\partial x_j}\frac{\partial\psi}{\partial x_j} \right\} dx,$$

$$\xi = \|\text{grad}\, w\|_2.$$

If $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ denote the vectors grad φ and grad ψ, respectively, and A denotes the matrix

$$\left\{ \frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j} \right\}_{i,j=1}^{2},$$

then

$$\langle H^M(w)\varphi, \psi\rangle_M = \int_M \boldsymbol{\varphi}^T B \boldsymbol{\psi}\, dx, \quad B = \alpha I + \beta A, \tag{4.2}$$

$$\alpha = \nu_M(\xi), \quad \beta = \xi^{-1}\nu_M'(\xi).$$

(I is the unit matrix). We easily find the eigenvalues of B:

$$\lambda_{1,2} = \begin{cases} \alpha + \frac{1}{2}(\beta + |\beta|)\xi^2, \\ \alpha + \frac{1}{2}(\beta - |\beta|)\xi^2. \end{cases}$$

If $\nu'(\xi) < 0$, then

$$\lambda_{1,2} = \begin{cases} \alpha = \nu_M(\xi), \\ \alpha + \beta\xi^2 = [\xi\nu_M(\xi)]'; \end{cases}$$

if $\nu'(\xi) \geqslant 0$ then

$$\lambda_{1,2} = \begin{cases} [\xi\nu_M(\xi)]', \\ \nu_M(\xi). \end{cases}$$

From (2.4) it follows that

$$c_M \leqslant \nu_M(\xi) \leqslant C_M \quad \forall \xi \in [0, \infty), \tag{4.3}$$

and we easily get that $\|B\|_2 = \max|\lambda_i| \leqslant C_M$. From (4.2) follows the first inequality in (2.3). Concerning the second inequality it is also true because

$$\langle H^M(w)\varphi, \varphi\rangle_M = \int_M \left\{ \nu_M(\xi)\|\text{grad}\,\varphi\|_2^2 + \xi^{-1}\nu_M'(\xi) \left[\sum_{i=1}^{2} \frac{\partial w}{\partial x_i}\frac{\partial \phi}{\partial x_i} \right]^2 \right\} dx,$$

and the integrand is bounded from below by

$$\nu_M(\xi)\|\text{grad}\,\varphi\|_2^2 \geqslant c_M\|\text{grad}\,\varphi\|_2^2 \quad \text{if } \nu' \geqslant 0$$

and by

$$[\nu_M(\xi) + \xi^{-1}\nu_M'(\xi)\xi^2]\,\|\text{grad}\,\varphi\|_2^2 \geqslant c_M\|\text{grad}\,\varphi\|_2^2 \quad \text{if } \nu' < 0.$$

The third inequality in (2.3) is obvious. In three dimensions we easily find that

$$\langle H^M(w)\varphi, \psi\rangle = \int_M \{\nu_M(\xi)(\operatorname{curl}\psi)^T \operatorname{curl}\varphi + \xi^{-1}\nu'_M(\xi) \times [(\operatorname{curl}\mathbf{w})^T\operatorname{curl}\varphi][(\operatorname{curl}\mathbf{w})^T\operatorname{curl}\psi]\}\,dx,$$

where $\xi = \|\operatorname{curl}\mathbf{w}\|_2$, and the proof of (2.3) is similar to that given above.

We will assume that $J \in C^{0,1}([0,T]; L^2(\Omega))$ and $u_0 = g|_R$ where $g \in H_0^1(\Omega)$. Further, let us consider a regular family of triangulations which consist of triangles belonging either to $\bar{R}$ or to $\bar{S}$, and let us take, for simplicity, piecewise linear functions (belonging to $C(\bar{\Omega}) \cap H_0^1(\Omega)$) as trial functions. The approximation g^h is determined according to E. Obviously, the conditions C, D, are also satisfied. The approximate solution U^δ is determined by (3.7) and by the equations

(4.4) $$(\Delta U^1, \sigma v)_{L^2(R)} + \Delta t a(U^1, v) = \Delta t(J^1, v)_{L^2(\Omega)} \quad \forall v \in V^h,\ U^0|_R = g^h|_R,$$

(4.5) $$(\Delta U^i, \sigma v)_{L^2(R)} + \Delta t l(U^i, v) = \Delta t\omega(U^{i-1}, v) + \Delta t(J^{i-1}, v)_{L^2(\Omega)} \quad \forall v \in V^h,\ i \geq 2,$$

where again $\omega(u, v) = l(u, v) - a(u, v)$,

$$a(u, v) = \sum_M \int_M \nu_M(\|\operatorname{grad} u\|_2) \sum_{i=1}^2 \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,dx,$$

$$l(u, v) = \sum_M \Theta_M \int_M \sum_{i=1}^2 \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,dx,$$

and we assume Θ_M to satisfy (3.1). From Theorem 3.1 we get

THEOREM 4.1. *Under the above conditions we have*

$$\|u - U^\delta\|_{C([0,T]; L^2(R))} \to 0, \quad \|u - U^\delta\|_{L^2(0,T; H_0^1(\Omega))} \to 0.$$

Now we derive error estimates under the condition that the exact solution is smooth in R and in S. For the initial value $U^0|_R$ we can take u_0 or any approximation u_0^h such that $\|u_0 - u_0^h\|_{L^2(R)} \leq Ch$.

THEOREM 4.2. *Besides the above conditions we assume that*

$$u|_M \in C([0,T]; H^2(M)), \quad M = R, S, \quad u' \in L^2(0,T; H^1(\Omega)),$$
$$u''|_R \in L^2(0,T; \bar{V}'_R).$$

Then we have

(4.6) $$\left\{\Delta t \sum_{i=1}^r \|u(t_i) - U^i\|^2_{H_0^1(\Omega)}\right\}^{1/2} = O(h + \Delta t).$$

Proof. First, we estimate the Hessians K^M of the functionals

$$\tfrac{1}{2}\Theta_M\|\varphi\|^2_{1,M} - J^M(\varphi) = \int_M \left[\tfrac{1}{2}\Theta_M\|\operatorname{grad}\varphi\|_2^2 - F^M(\|\operatorname{grad}\varphi\|_2)\right]dx.$$

We have

$$\langle K^M(w)\varphi, \psi\rangle_M = \int_M \varphi^T D\psi\,dx, \qquad D = \Theta_M I - B.$$

The eigenvalues μ of D are of the form $\mu = \Theta_M - \lambda$, where the λ's are eigenvalues of B. As $c_M \leqslant \lambda \leqslant C_M$ (see above) and $\Theta_M > \frac{1}{2}C_M$, one can easily prove that

$$|\mu| \leqslant \Theta_M - \gamma_M, \qquad \gamma_M = \min\left(c_M, 2\left(\Theta_M - \tfrac{1}{2}C_M\right)\right), \qquad 0 < \gamma_M < \Theta_M. \tag{4.7}$$

Hence

$$\left|\left\langle K^M(w)\varphi, \psi\right\rangle_M\right| \leqslant \tau_M \|\varphi\|_{1,M}\|\psi\|_{1,M}, \qquad 0 < \tau_M = \Theta_M - \gamma_M < \Theta_M$$

and

$$\left|\omega^M(\varphi, w) - \omega^M(\psi, w)\right| \leqslant \tau_M\|\varphi - \psi\|_{1,M}\|w\|_{1,M} \quad \forall \varphi, \psi, w \in \bar{V}_M. \tag{4.8}$$

We denote by $\hat{u}^i$ a modified Clément approximation of $u(t_i)$ (see [9, Section 4]). For $i \geqslant 2$ we have

$$\begin{aligned}
&(\Delta\hat{u}^i, \sigma v)_{L^2(R)} + \Delta t l(\hat{u}^i, v)\\
&\qquad = \Delta t \omega(\hat{u}^{i-1}, v) + \Delta t(J^{i-1}, v)_{L^2(\Omega)} + \Delta t(\Delta J^i, v)_{L^2(\Omega)}\\
&\qquad\quad + \left(\Delta u^i - \Delta t u'(t_i), \sigma v\right)_{L^2(R)} + \left(\Delta(\hat{u}^i - u^i), \sigma v\right)_{L^2(R)}\\
&\qquad\quad + \Delta t\left[\omega(\hat{u}^i, v) - \omega(\hat{u}^{i-1}, v)\right] + \Delta t\left[a(\hat{u}^i, v) - a(u^i, v)\right].
\end{aligned}$$

If we prove that $\Delta t \Sigma_{i=1}^r \|\varepsilon^i\|_1^2 = O(h^2 + \Delta t^2)$ where $\varepsilon^i = \hat{u}^i - U^i$, then (4.6) follows by means of Lemma 3 of [9]. Subtracting (4.5) from the above equation and choosing $v = \varepsilon^i$, we get

$$\begin{aligned}
&(\Delta\varepsilon^i, \sigma\varepsilon^i)_{L^2(R)} + \Delta t\left\{l(\varepsilon^i, \varepsilon^i) - \left[\omega(\hat{u}^{i-1}, \varepsilon^i) - \omega(U^{i-1}, \varepsilon^i)\right]\right\}\\
&\quad = \left(\Delta u^i - \Delta t u'(t_i), \sigma\varepsilon^i\right)_{L^2(R)} + \left(\Delta(\hat{u}^i - u^i), \sigma\varepsilon^i\right)_{L^2(R)}\\
&\qquad + \Delta t\left[a(\hat{u}^i, \varepsilon^i) - a(u^i, \varepsilon^i)\right] + \Delta t(\Delta J^i, \varepsilon^i)_{L^2(\Omega)} + \Delta t\left[\omega(\hat{u}^i, \varepsilon^i) - \omega(\hat{u}^{i-1}, \varepsilon^i)\right].
\end{aligned}$$

By (4.8) the second term on the left-hand side is bounded from below by

$$\begin{aligned}
&\Delta t \sum_M \left\{\Theta_M\|\varepsilon^i\|_{1,M}^2 - \tfrac{1}{2}\tau_M\left[\|\varepsilon^{i-1}\|_{1,M}^2 + \|\varepsilon^i\|_{1,M}^2\right]\right\}\\
&\quad \geqslant \tfrac{1}{2}\gamma_0 \Delta t\|\varepsilon^i\|_1^2 + \tfrac{1}{2}\Delta t \sum_M \left[\Theta_M\|\varepsilon^i\|_{1,M}^2 - \tau_M\|\varepsilon^{i-1}\|_{1,M}^2\right], \qquad \gamma_0 = \min(\gamma_R, \gamma_S) > 0.
\end{aligned}$$

Therefore, if we sum up from $i = 2$ to $i = r$, we find that the left-hand side is bounded from below by

$$-\tfrac{1}{2}(\varepsilon^1, \sigma\varepsilon^1)_{L^2(R)} + \tfrac{1}{2}\gamma_0\Delta t \sum_{i=2}^r \|\varepsilon^i\|_1^2 - \tfrac{1}{2}\tau_1\Delta t\|\varepsilon^1\|_1^2, \qquad \tau_1 = \max(\tau_R, \tau_S).$$

The first three terms of the right-hand side are equal to the right-hand side of the equation (4.21) in [9]. After summing up we get as in [9] the following bound from above:

$$\tfrac{1}{8}\gamma_0\Delta t \sum_{i=2}^r \|\varepsilon^i\|_1^2 + C(h^2 + \Delta t^2).$$

Here, the constant C depends on u. The same result is true for the remaining terms of the right-hand side. It follows that

$$\Delta t \sum_{i=2}^r \|\varepsilon^i\|_1^2 \leqslant C\left\{\|\varepsilon^1\|_{L^2(R)}^2 + \Delta t\|\varepsilon^1\|_1^2 + h^2 + \Delta t^2\right\}. \tag{4.9}$$

For $i = 1$ we have

$$(\Delta\hat{u}^1, \sigma v)_{L^2(R)} + \Delta t a(\hat{u}^1, v)$$
$$= (\Delta u^1 - \Delta t u'(t_1), \sigma v)_{L^2(R)} + (\Delta(\hat{u}^1 - u^1), \sigma v)_{L^2(R)}$$
$$+\Delta t[a(\hat{u}^1, v) - a(u^1, v)] + \Delta t(J^1, v)_{L^2(\Omega)}.$$

In a similar way we obtain

$$(\Delta\varepsilon^1, \sigma\varepsilon^1)_{L^2(R)} + \Delta t[a(\hat{u}^1, \varepsilon^1) - a(U^1, \varepsilon^1)]$$
$$= (\Delta u^1 - \Delta t u'(t_1), \sigma\varepsilon^1)_{L^2(R)} + (\Delta(\hat{u}^1 - u^1), \sigma\varepsilon^1)_{L^2(R)}$$
$$+\Delta t[a(\hat{u}^1, \varepsilon^1) - a(u^1, \varepsilon^1)].$$

By (3.10) and the assumption $\sigma \in L^\infty(R)$, $\sigma \geqslant \sigma_0 > 0$ we easily get

$$\sigma_0\|\varepsilon^1\|^2_{L^2(R)} + c_0\Delta t\|\varepsilon^1\|^2_1 \leqslant C[\|\varepsilon^0\|^2_{L^2(R)} + h^2 + \Delta t^2] + \tfrac{1}{2}c_0\Delta t\|\varepsilon^1\|^2_1,$$

so that

$$\|\varepsilon^1\|^2_{L^2(R)} + \Delta t\|\varepsilon^1\|^2_1 = O(h^2 + \Delta t^2).$$

This together with (4.9) proves the theorem.

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